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Global Existence and Nonexistence Theorems for Quasilinear Evolution Equations of Formally Parabolic Type

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1. INTRODUCTION

Let H be a Hilbert space with inner product $(\cdot, \cdot)$ and let D_A, D_Q be densely defined subspaces on which self-adjoint linear operators $Q: D_Q \to H$, $A: D_A \to H$ are given. Let D be a third subspace of H and let F be a gradient operator from D into its dual D' i.e., there is a potential $\mathscr{F}: D \to C$ (or R) such that

$$\mathscr{F}'(x) \cdot y = (F(x), y)_D,$$

where $\mathscr{F}'$ denotes the Fréchet derivative of $\mathscr{F}$ and $(\ ,\ )_D$ is the natural pairing in D. Let $(Qx, x) > 0$, $(Ax, x) \geqslant 0$ for $x \neq 0$ in the appropriate operator domain. Let $J = [0, \infty)$. In [9] the problem

$$Qu_t = -Au + F(u), \tag{1.1}$$

$$u(0) = u_0 \tag{1.2}$$

was considered with $u(t) \in D \cap D_A$ and $u_t(t) \in D_Q$ for $t > 0$. It was shown that if the initial energy for (1.1) were negative, i.e.,

$$E(0) \equiv \tfrac{1}{2}(u_0, Au_0) - \mathscr{F}(u_0) < 0,$$

if the energy inequality

$$E(t) \equiv \tfrac{1}{2}(u(t), Au(t)) - \mathscr{F}(u(t)) \leqslant E(0) - \int_0^t (Qu_t, u_t)\, d\eta \tag{1.3}$$

were satisfied along solutions, and if there were a number $\alpha > 0$ such that

$$(F(x), x) \geqslant (2\alpha + 2)\, \mathscr{F}(x)$$

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for all $x \in D$, then (1.1)–(1.2) could not have a solution on J. (Solutions were first assumed to be classical so that equality actually holds in (1.3). However, a version of this result was also given for weak solutions, for which the inequality was assumed.)

Other applications were also given to systems of parabolic equations and to parabolic equations with *nonlocal* nonlinearities.

Prior to this result, blow up (global nonexistence) proofs for semilinear parabolic equations were usually given by an argument of Kaplan [6] or by the method of subsolutions.

In this paper, we consider a more general version of (1.1), namely

$$Q(t, u, u_t) + A(t, u) = F(t, u), \tag{1.4}$$

where now A, F, Q are nonlinear operators on appropriate Banach spaces. In this formulation, we understand A as an appropriate differential operator and F as the driving force. Some special cases have already been treated in the literature. For example, when $Q(t, u, u_t) = u_t$, $A(u) = \Delta\phi(u)$, and $F(u) = u^p$, the initial value problem for (1.4) has been studied by many authors; see [16]. Moreover, when $Q(t, u, u_t) = a'(u) u_t$, $A(u) = \Delta u$, and $F(u) = u^p$ the equation (1.4) for nonnegative initial values (either in bounded domains with homogeneous boundary conditions or in all of R^N) reduces, at least formally, to some of the problems considered in [16].

The purpose of this paper is, first, to formulate a precise definition for the meaning of Eq. (1.4). In this we shall follow and generalize previous work of Nakao [14] and Pucci and Serrin [15]. We will then improve and extend to the abstract equation (1.4) the argument used in [12, 13] to prove global nonexistence. In particular, we allow strongly non-autonomous behavior for the operator Q, and we show, under appropriate conditions on the time behavior of Q, that once the energy becomes at all negative, the solution *cannot* exist for all time (see Theorem 1 in Section 3). Note particularly that although the operator $A(\cdot)$ need not be linear, it must be derivable from a potential, just as was the case for the concavity method used in [9, 10].

In Section 4 we give a global existence theorem for the initial value problem for (1.4) by assuming a structure condition on F which is, in some sense, dual to that needed for global nonexistence. In Sections 5 we apply these abstract results to the first initial–boundary value problem for

$$|u_t|^{m-2} u_t - a\nabla \cdot (|\nabla u|^{q-2} \nabla u) = c\, |u|^{p-2} u$$

and, with some modification, to the same problem for equations of the form

$$|u|^{\kappa}\, |u_t|^{m-2} u_t - a\nabla \cdot (|\nabla u|^{q-2} \nabla u) = c\, |u|^{p-2} u.$$

In the case $\kappa = m-2 = 0$, a global nonexistence result for solutions of the first initial–boundary value problem can be obtained for solutions of this last equation using the aforementioned concavity argument [11]. However, when $m > 2$, concavity arguments appear to be inapplicable to this equation.

Equation (1.4) contains, as a special case, the porous medium equation,

$$u_t = \Delta(|u|^{\ell-1} u) + |u|^{s-1} u, \tag{1.5}$$

when $0 < \ell < 1$ and $s > 1$. It is known that, for this equation, *global* solutions on $\Omega \times J$ with *negative initial energy*

$$\mathscr{E}(0) = \frac{1}{2l} \int |\nabla(|u_0|^{\ell-1} u_0)|^2 \, dx - \frac{1}{l+s} \int |u_0|^{\ell+s} \, dx \tag{1.6}$$

cannot exist.

This result was proved by Galaktionov [4] in the slow diffusion case ($l > 1$) but his proof is also valid for fast diffusion ($0 < l < 1$). See also [16, p. 210], as well as [7], where global nonexistence was proved for sufficiently large initial values for problems in bounded domains with homogeneous boundary conditions (Dirichlet, Neumann, or mixed) using a modification of an argument of Kaplan [6]). It is perhaps worth remarking that the concavity argument can also be used to obtain global nonexistence of weak solutions with negative initial energy; see [2]. The functional used there must, however, be modified somewhat. One uses

$$F(t) = \int_0^t \int_\Omega |u|^{l+1} \, dx \, d\eta + (T-t) \int_\Omega |u_0|^{l+1} \, dx + \beta(t+t_0)^2,$$

formulates the problem in an appropriate weak sense, and then proceeds along the lines of [9], choosing β, t_0, and T appropriately. In [2] only the first term was needed because the author wished to show that if a weak solution were global, then the Dirichlet norm of u^l could not remain bounded.

Our results can also be applied to the general equation

$$Q(t, x, u, u_t) - D \cdot (a(x, t) \, |Du|^{q-2} \, Du) = f(x, t, u),$$

although we leave the details to the reader.

The proofs of global nonexistence in the literature do not, in general, imply finite time blow-up of the solution itself. On the other hand, if one can couple global nonexistence with a local continuation argument based on the assumption of an appropriate a priori bound for the solution, as in

[5], then global nonexistence will imply finite time blow-up. We take this up in Section 3, following the statement of Theorem 1.

2. PROBLEM FORMULATION AND DEFINITIONS

We begin with some preliminary notation and concepts. Throughout the remainder of the paper, we let $J = J_\infty = [0, \infty)$. Let X be a (possibly) complex Banach space, and X' its dual. We let $\langle \cdot,\cdot \rangle_X$ be the natural pairing between X and X', so that if $x \in X$, $x' \in X'$ then

$$\langle x', x \rangle_X = x'(x) \in C.$$

Of course $|\langle x', x \rangle_X| \leqslant \|x\|_X \cdot \|x'\|_{X'}$. Let $\mathscr{F} \in C^1(J \times X \to C)$ with $\mathscr{F}(t, 0) = 0$, and, for $t \in J$, let $F(t, \cdot)$ be the Fréchet derivative of $\mathscr{F}(t, \cdot)$. It is easy to see that the representation

$$\mathscr{F}(t, x) = \int_0^1 \langle F(t, \tau x), x \rangle_X \, d\tau$$

holds for all $x \in X$. From this and the definition of the derivative, it follows that $\mathscr{F}$ is real valued if and only if $\langle F(t, x), x \rangle_X$ is real valued.

Now let D, X, Y be Banach spaces with respective norms $\|\cdot\|_D$, $\|\cdot\|_X$, $\|\cdot\|_Y$, and pairings $\langle \cdot, \cdot \rangle_D$, $\langle \cdot, \cdot \rangle_X$, $\langle \cdot, \cdot \rangle_Y$. For simplicity in printing in the remainder of the paper, the pairing subscripts D, X, Y will be omitted whenever the meaning is clear from the context. For our purposes, it is assumed moreover that D, X, and Y have a common subspace Z.

We are given two functions

$$A : J \times D \to D', \qquad F : J \times X \to X'.$$

Associated with A and F, respectively, are C^1 potential functions

$$\mathscr{A} : J \times D \to R, \qquad \mathscr{F} : J \times X \to R$$

such that $\mathscr{A}(t, 0) = \mathscr{F}(t, 0) = 0$ and

$$\mathscr{A}_u(t, u) = A(t, u), \qquad \mathscr{F}_u(t, u) = F(t, u).$$

Finally, let S be a given subset of $J \times Y$ and Q a continuous mapping from S into X'.

We consider the abstract evolution problem

$$Q(t, u_t) + A(t, u(t)) = F(t, u(t)), \qquad t \in J \tag{2.1}$$

$$u(0) = u_0, \tag{2.2}$$

where $u_0 \in Z$. (The slightly more general equation where $Q(t, u(t), u_t)$ replaces $Q(t, u_t)$ will be taken up at the end of Section 3.) A (strong) solution of (2.1) is understood to be a function $u \in K$ where

$$K = \{\varphi : J \to Z \mid \varphi \in C(J \to D) \cap C(J \to X) \cap AC(J \to Y)\},$$

which, to begin with, satisfies the *distribution relation*

$$0 = \int_0^t \{\langle A(\eta, u), \varphi\rangle + \langle Q(\eta, u_t), \varphi\rangle - \langle F(\eta, u), \varphi\rangle\}\, d\eta \tag{2.3}$$

for all $\varphi \in K$ and $t \in J$; see [15]. (Here $AC(J \to V)$ denotes the set of absolutely continuous functions from J into V.)

Remark 1. If we write $a(t, u, v)$ for the quantity $\langle A(t, u), v\rangle$, then it is clear that

$$a : J \times D \times D \to R$$

and that a is linear in its last variable. From an alternative point of view, it is precisely the function $a(t, u, v)$ which defines the element $A(t, u)$ in D'.

For example, in a concrete problem, say when $A(t, u) = -\Delta u$ and $D = H_0^1(\Omega)$, we have

$$\langle A(t, u), v\rangle = a(t, u, v) = \int_\Omega \nabla u \cdot \nabla v\, dx$$

and

$$\mathscr{A}(t, u) = \tfrac{1}{2} \int_\Omega |\nabla u|^2\, dx.$$

Other specific examples are considered in Section 5.

Let $\mathscr{E}$ be the *potential energy* of a field $u \in K$; that is,

$$\mathscr{E}(t) = \mathscr{A}(t, u) - \mathscr{F}(t, u). \tag{2.4}$$

Obviously $\mathscr{E}(\cdot) \in C(J)$. An additional, and crucial, element in the definition of a solution is an appropriately formulated conservation law for the energy. If we were to proceed purely formally, we would obtain the strict conservation relation

$$\mathscr{E}(t) = \mathscr{E}(0) - \int_0^t [\langle Q(\eta, u_t(\eta)), u_t(\eta)\rangle - \mathscr{A}_t(\eta, u(\eta)) + \mathscr{F}_t(\eta, u(\eta))]\, d\eta.$$

For our present purposes this is stronger than necessary; instead we postulate the following *weak conservation law*.

There exists a function $\mathscr{D}(t, y)$ from $J \times Y$ into the extended reals, called the *dissipation rate*, such that for any solution $u \in K$ we have

$$\mathscr{E}(t) + \int_0^t [\mathscr{D}(\eta, u_t(\eta)) - \mathscr{A}_t(\eta, u(\eta)) + \mathscr{F}_t(\eta, u(\eta))] \, d\eta \leqslant \mathscr{E}(0), \quad (2.5)$$

for $t \in J$, with $\mathscr{D}(t, u_t)$ *being locally integrable* on J.

We assume that the dissipation rate has the following structure:

(1d) $\mathscr{D}(t, y) < \infty$ implies $(t, y) \in S$

(2d) $\mathscr{D}(t, y) \geqslant 0$ for $(t, y) \in S$.

(3d) there is an exponent $m > 1$ and a nonnegative locally bounded function $\delta(t)$ such that

$$\|Q(t, y)\|_{X'} \leqslant [\delta(t)]^{1/m} [\mathscr{D}(t, y)]^{1/m'} \qquad \text{on } S, \quad (2.6)$$

where m' is the Hölder conjugate of m.

Since for any solution the function $\mathscr{D}(t, u_t)$ is assumed to be locally integrable on J, condition (1d) shows that

$$(t, u_t) \in S \qquad \text{for a.a.} \quad t \in J.$$

In turn, the quantity $Q(t, u_t)$ in (2.3) is well defined for almost all $t \in J$.

We next show that the distribution identity itself is meaningful. Indeed, the terms $\langle A, \varphi \rangle$, $\langle F, \varphi \rangle$ in (2.3) are obviously well defined continuous functions of t. From (2.6), moreover, the term $\langle Q(t, u_t), \varphi \rangle$ is locally integrable. Indeed, for almost all $t \in J$,

$$|\langle Q, \varphi \rangle| \leqslant \|Q\|_{W'} \|\varphi\|_X \leqslant [\delta(t)]^{1/m} [\mathscr{D}(t, u_t)]^{1/m'} \|\varphi(t)\|_X,$$

so by Hölder's inequality

$$\int_0^t |\langle Q, \varphi \rangle| \, d\eta \leqslant \left(\int_0^t \delta(\eta) \, d\eta \right)^{1/m} \left(\int_0^t \mathscr{D}(\eta, u_t) \, d\eta \right)^{1/m'} \cdot \sup_{[0, t]} \|\varphi(\eta)\|_X < \infty,$$

as required (use the fact that φ is continuous from J into X).

The exponent m and the function $\delta(t)$ in (3d) essentially determine the degree of non-linearity and the degree of non-autonomy of the damping term Q, as can be seen from the following canonical example (see [15, Sect. 2]). Take $S = J \times Y$, with X being continuously embedded in Y, and $Q : S \to Y'$. Suppose

(a) $\|Q(t, y)\|_{Y'} \leqslant \delta(t) \|y\|_Y^{m-1}$ together with the reverse pairing inequality,

(b) $\|Q(t, y)\|_{Y'} \|y\|_Y \leqslant \gamma \langle Q(t, y), y \rangle_Y$ with $\gamma = \text{constant} \geqslant 1$.

We choose

$$\mathscr{D}(t, y) = \langle Q(t, y), y \rangle_Y$$

in (2.5). Clearly (1d) and (2d) are satisfied. Moreover, from (a) and (b)

$$\begin{aligned} \|Q(t, y)\|_{Y'} &= \|Q(t, y)\|_{Y'}^{1/m} \|Q(t, y)\|_{Y'}^{1/m'} \\ &\leqslant [\delta(t)]^{1/m} \|y\|_Y^{1/m'} \cdot [\gamma \langle Q(t, y), y \rangle_Y / \|y\|_Y]^{1/m'} \\ &= [\gamma^{m-1} \delta(t)]^{1/m} [\mathscr{D}(t, y)]^{1/m'}, \end{aligned}$$

which implies (2.6) up to an inessential factor, since $\|u\|_{X'} \leqslant \text{Const} \cdot |u|_{Y'}$. (The factor γ can be allowed to depend on time, without any difficulty.)

For $T > 0$, let $J_T = [0, T)$. We conclude the section with the following local existence and continuation hypothesis (H), which will be of importance in Section 4:

H(i) *Local existence.* Whenever $u_0 \in Z$, there is a number $T \equiv T(u_0) \in (0, \infty)$, and a function

$$u \in K_T \equiv \{\varphi : J_T \to Z \mid \phi \in C(J_T \to D) \cap C(J_T \to X) \cap AC(J_T \to Y)\}$$

which solves (2.1)–(2.2) on J_T in the weak sense above.

H(ii) *Continuation.* If a local solution of (2.1)–(2.2) exists on an interval J_T and is bounded there in modified energy norm, that is,

$$\sup_{J_T} (|\mathscr{A}(t, u)| + |\mathscr{F}(t, u)|) < \infty,$$

then it can be continued as a solution to a larger interval $J_{T'}$ with $T' > T$.

Remark 2. There are situations for which the modified energy norm remains bounded but the solution cannot be continued, as can be shown using the results of [3, 8]. Roughly, the idea is the following: Let $\Omega = B(0, R)$ denote the ball of radius R in R^n with center at the origin and let the exponent p be *supercritical*, namely $p > 2n/(n-2)$. Consider the initial–boundary value problem

$$\begin{aligned} u_t &= \Delta u + u^{p-1} \qquad && (x, t) \in \Omega \times J, \\ u &= 0 && (x, t) \in \partial\Omega \times J, \\ u(x, 0) &= u_0(x) > 0 && x \in \Omega. \end{aligned}$$

It is possible by [3] to construct positive, radially decreasing initial values with the following properties: (i) The solution u blows up in finite time T say, in the supremum norm; (ii) the solution has negative initial energy; and (iii) the $\mathscr{L}^p$ norm remains finite. Consequently the modified energy must remain bounded. Then, using [8] and the Maximum Principle, one can construct a somewhat larger initial function with these same properties *for which the solution blows up completely* (*at every point of* Ω) in finite time $T' < T$. Such a solution cannot be continued in any meaningful sense beyond T'.

3. GLOBAL NONEXISTENCE RESULTS

For our principal results, we make use of the following structural hypotheses for the functions A, F:

(1s) $\mathscr{A}(t, u) \geqslant 0$, and there is a positive constant q such that $q\mathscr{A}(t, u) \geqslant \langle A(t, u), u \rangle$ for all $(t, u) \in J \times Y$.

(2s) There is a positive constant $c = c(\tau)$, defined for each $\tau > 0$, and an exponent $p > 1$ such that

$$c\mathscr{F}(t, u) \, \|u\|_X^{p'} \leqslant \langle F(t, u), u \rangle - q\mathscr{F}(t, u)$$

whenever $(t, u) \in J \times X$ and $\mathscr{F}(t, u) \geqslant \tau$.

(3s) For all $(t, u) \in J \times Z$ for which $\mathscr{A}(t, u) < \mathscr{F}(t, u)$, we have

$$\mathscr{A}_t(t, u) - \mathscr{F}_t(t, u) \leqslant 0. \tag{3.1}$$

We will need the following result.

LEMMA 1. *Assume* (3s). *Then, along any solution* u *of* (2.1)–(2.2) *with* $\mathscr{E}(0) < 0$,

$$\mathscr{E}(t) + \int_0^t \mathscr{D}(\eta, u_t(\eta)) \, d\eta \leqslant \mathscr{E}(0), \qquad t \in J. \tag{3.2}$$

Proof. We first show that, along any solution u of (2.1)–(2.2) with $\mathscr{E}(0) < 0$,

$$\mathscr{A}(t, u(t)) - \mathscr{F}(t, u(t)) < 0 \tag{3.3}$$

for all $t \in J$. Indeed, since $\mathscr{E}(0) < 0$, we have

$$\mathscr{A}(0, u(0)) - \mathscr{F}(0, u(0)) < 0.$$

If (3.3) fails to hold for all $t \in J$, then there exists a first vaolue $T > 0$ such that

$$\mathscr{A}(T, u(T)) - \mathscr{F}(T, u(T)) = 0.$$

Consequently by (3s),

$$\mathscr{A}_t(t, u(t)) - \mathscr{F}_t(t, u(t)) \leqslant 0, \qquad t \in [0, T].$$

Then by the energy conservation law (2.5), since $\mathscr{D}$ is non-negative,

$$\mathscr{E}u(t) \leqslant \mathscr{E}u(0) < 0, \qquad t \in [0, T].$$

Hence

$$\mathscr{A}(T, u(T)) - \mathscr{F}(T, u(T)) < 0,$$

which is the required contradiction.

Consequently (3.3) is also valid, and in turn by (3s), condition (3.1) also holds. Then in view of (2.5), we obtain (3.2).

COROLLARY. *The assertion of Lemma* 1 *remains true if* (3s) *is replaced by the property that*

$$(t, u) \in J \times Z \qquad \textit{and} \qquad \mathscr{F}(t, u) > 0 \tag{3.4}$$

implies

$$\mathscr{F}_t(t, u) \geqslant 0 \qquad \textit{and} \qquad (\mathscr{A}/\mathscr{F})_t\,(t, u) \leqslant 0. \tag{3.5}$$

Proof. We show that the above property implies (3.1) when $\mathscr{A} < \mathscr{F}$. Thus let $(t, u) \in J \times Z$ be such that $\mathscr{A}(t, u) < \mathscr{F}(t, u)$. Then $\mathscr{F}(t, u) > 0$ and so (3.5) can be applied; that is,

$$\mathscr{A}_t(t, u) - \mathscr{F}_t(t, u) \leqslant \frac{\mathscr{F}_t(t, u)}{\mathscr{F}(t, u)} \{\mathscr{A}(t, u) - \mathscr{F}(t, u)\} \leqslant 0.$$

Special Case. $\mathscr{A}(t, u) = a(t)\, \hat{\mathscr{A}}(u)$ and $\mathscr{F}(t, u) = f(t)\, \hat{\mathscr{F}}(u)$. If $a \geqslant 0$ and $f > 0$ on J, and $\hat{\mathscr{A}} \geqslant 0$ on Z, then (3.5) holds provided $(a/f)' \leqslant 0$ and $f' \leqslant 0$ on J.

THEOREM 1. *Suppose that*

$$1 < m < p; \tag{3.6}$$

that the structure conditions (1d)–(3d), (1s)–(3s) *hold; and that*

$$\int^{\infty} \delta^{-1/(m-1)}(t)\, dt = \infty. \tag{3.7}$$

Then no solution of (2.1)–(2.2) *with* $\mathscr{E}(0) < 0$ *can exist on* J.

Remark 3. If hypothesis H(ii) in Section 2 is in force, then the conclusion can be recast in the form: Every solution of (2.1)–(2.2) with $\mathscr{E}(0) < 0$ must blow up in finite time T in the sense that $\lim_{t \to T^-} \mathscr{A}(t, u(t)) = \infty$. (Notice that $\mathscr{F} \geqslant \mathscr{A}$ along solutions with negative initial energy.)

Proof. Assume for contradiction that there is a solution of (2.1)–(2.2) on J. Define

$$\mathscr{H}(t) = \int_0^t \mathscr{D}(\eta, u_t(\eta))\, d\eta - \mathscr{E}(0). \tag{3.8}$$

In view of the condition $\mathscr{E}(0) < 0$, the assumption (1s) and the relations (2.4), (3.2), (3.8), (2d) we observe at the outset that

$$\mathscr{F}(t, u(t)) \geqslant -\mathscr{E}(t) \geqslant \mathscr{H}(t) \geqslant \tau > 0, \tag{3.9}$$

where $\tau \equiv -\mathscr{E}(0)$. To simplify the notation in the remainder of the proof, we shall frequently suppress various arguments in the relations that follow.

Putting $\varphi = u$ in (2.3) and using (2.4) and (3.9) now gives, a.e. in J,

$$\begin{aligned} 0 &= \langle F(t, u) - A(t, u) - Q, u \rangle + q(\mathscr{A}(t, u) - \mathscr{F}(t, u) - \mathscr{E}(t)) \\ &\geqslant (q\mathscr{A}(t, u) - \langle A(t, u), u \rangle) + (\langle F(t, u), u \rangle - q\mathscr{F}(t, u)) - \langle Q, u \rangle. \end{aligned}$$

Hence from (1s)–(3s) it follows that

$$\|Q\|_{X'}\, |u|_X \geqslant \langle Q, u \rangle \geqslant c\mathscr{F}^{1/p'} \|u\|_X.$$

Then from (3.9) again,

$$\|Q\|_{X'} \geqslant c\mathscr{F}^{1/p'} \geqslant c\mathscr{H}^1/p'.$$

On the other hand, since $\mathscr{D} = \mathscr{H}'$, we obtain from (3d)

$$\|Q\|_{X'}^{m'} \leqslant \delta^{1/(m-1)} \mathscr{H}'$$

so that elimination of $\|Q\|_X$ yields

$$\mathscr{H}' \geqslant c^{m'} \delta^{-1/(m-1)} \mathscr{H}^{m'/p'}. \tag{3.10}$$

However,

$$\frac{m'}{p'} - 1 = m' \left(\frac{1}{p'} - \frac{1}{m'} \right) = m' \left(\frac{1}{m} - \frac{1}{p} \right) > 0.$$

It follows from this observation, together with (3.7) and a quadrature, that $\mathscr{H}$ must blow up in finite time. Hence the supposition that u exists on the entire interval J must be false.

THEOREM 2. *Suppose that*

$$1 < m < p, \tag{3.11}$$

that the structure conditions (1d)–(3d) *and* (1s)–(3s) *hold, and that*

$$I \equiv \int_0^\infty \delta^{-1/(m-1)}(t)\, dt < \infty. \tag{3.12}$$

Then no solution of (2.1)–(2.2) *with* $\mathscr{E}(0) \ll 0$ *can exist in* $[0, \infty)$.

Proof. This follows directly from (3.10) after a quadrature. The precise condition is

$$\mathscr{E}(0) < -\max \left\{ 1, \left(\frac{p'}{(m'-p')\, c^{m'} I} \right)^{p'/(m'-p')} \right\}, \tag{3.13}$$

which is automatically satisfied if $I = \infty$. To see this, notice that if $\mathscr{E}(0) < -1$, then c can be fixed for the value $\tau = 1$.

Remark 4. Some additional generality can readily be achieved by allowing Q to depend on the solution as well as on its time derivative. Specifically, suppose that instead of (2s), (3d) we have:

(2s)′ There are positive constants $c_i = c_i(\tau)$, $i = 1, 2$, defined for each $\tau > 0$, and an exponent $p > 1$ such that

$$c_1(\tau)\, \mathscr{F}(t, u) \leqslant c_2(\tau)\, \|u\|_X^p \leqslant \langle F(t, u), u \rangle - q\mathscr{F}(t, u)$$

whenever $(t, u) \in J \times X$ and $\mathscr{F}(t, u) \geqslant \tau$.

(3d)′ There is an exponent $m > 1$, a real number κ, and a non-negative locally bounded function $\delta(t)$ such that

$$\|Q(t, u, v)\|_{X'} \leqslant \{\delta(t)[\, \|u\|_X^\kappa + 1]\}^{1/m}\, [\mathscr{D}(t, u, v)]^{1/m'} \quad \text{on } S, \tag{3.14}$$

where now $0 \leqslant \kappa < p - m$.

Theorems 1 and 2 continue to hold. To see this, we use (2s)′ and (3.14) to obtain, in place of (3.10), the differential inequality

$$\mathscr{H}' \geqslant c^{m'} \delta^{-1/(m-1)} \frac{\mathscr{H}^{m'/p'}}{\mathscr{H}^{\mu}+1},$$

where $\mu \equiv \kappa/(m-1)p$. Hence, since $\mathscr{H}$ is increasing,

$$\mathscr{H}' \geqslant \text{Const. } \delta^{-1/(m-1)} \mathscr{H}^{m'/p'-\mu},$$

for an appropriate constant (depending on $|\mathscr{E}(0)|$). A short computation shows that

$$\frac{m'}{p'} - \mu - 1 = \frac{p-m-\kappa}{p(m-1)} > 0,$$

and hence $\mathscr{H}$ blows up in finite time. The analogue of (3.13) is easily written down by replacing m'/p' by $m'/p' - \mu$ in (3.10). Again the conclusions of Theorems 1 and 2 remain valid.

4. GLOBAL EXISTENCE RESULTS

We assume the following structure conditions on $\mathscr{A}$, $\mathscr{F}$ and $\mathscr{D}$:

(1e) $\mathscr{A}(t, u) \geqslant 0$, $\mathscr{F}(t, u) \geqslant 0$ for all $(t, u) \in J \times Z$.

(2e) There is a constant $d_1 < 1$ and functions $d_2, d_3 \in \mathscr{L}^1_{loc}(J)$ such that

$$\langle F(t, u), y \rangle \leqslant d_1 \mathscr{D}(t, y) + d_2(t)\, \mathscr{F}(t, u) + d_3(t) \tag{4.1}$$

for $t \in J$, $u \in Z$ and $y \in Y$.

(3e) Let $(\cdot)^+ = \max\{\cdot, 0\}$. The function

$$d_4(t) \equiv \sup_{u \in Z} \frac{\mathscr{A}_t^+(t, u) + \mathscr{F}_t^+(t, u)}{\mathscr{A}(t, u) + \mathscr{F}(t, u)}$$

is in $\mathscr{L}^1_{loc}(J)$.

Special Case. $\mathscr{A}(t, u) = a(t)\, \hat{\mathscr{A}}(u)$ and $\mathscr{F}(t, u) = f(t)\, \hat{\mathscr{F}}(u)$. If $a, f > 0$ on J, and $\hat{\mathscr{A}}, \hat{\mathscr{F}} \geqslant 0$ on Z, we then have $A(t, u) = a(t)\, \hat{A}(u)$ and $F(t, u) = f(t)\, \hat{F}(u)$, where $\hat{\mathscr{A}}_u = \hat{A}$ and $\hat{\mathscr{F}}_u = \hat{F}$. Moreover, (3e) will hold provided the following upper bound for $d_4(t)$,

$$\frac{(a'(t))^+}{a(t)} + \frac{(f'(t))^+}{f(t)}$$

is in $\mathscr{L}^1_{loc}(J)$. In addition, suppose $\mathscr{D}(t, y) = d(t)\,\hat{\mathscr{D}}(y)$ with $d > 0$ on J. In place of (4.1) we assume, for $u \in Z$ and $y \in Y$,

$$\langle \hat{F}(u), y \rangle \leqslant \hat{d}_1 \hat{\mathscr{D}}(y) + \hat{d}_2 \hat{\mathscr{F}}(u) + \hat{d}_3, \tag{4.2}$$

where the $\hat{d}_i$ are *constants* and where

$$\hat{d}_1 < \inf \left\{ \frac{f(t)}{d(t)} \,\middle|\, t \in J \right\}.$$

Then (4.1) holds with

$$d_1 \equiv \hat{d}_1 \sup \left\{ \frac{d(t)}{f(t)} \,\middle|\, t \in J \right\}, \qquad d_2(t) \equiv \hat{d}_2, \qquad d_3(t) \equiv \hat{d}_3\, f(t).$$

Finally, assume that the hypotheses H(i), H(ii) at the end of Section 2 hold for solutions of (2.1)–(2.2). Then we have the following global existence theorem:

THEOREM 3. *Let A, $\mathscr{D}$, F satisfy the structure conditions* (1e)–(3e) *above. Then for* every $u_0 \in Z$, *the problem* (2.1), (2.2) *has at least one global solution. Moreover, every solution corresponding to data $u_0 \in Z$ is global* (*i.e., every local solution can be extended to a global solution*).

Proof. Define the modified energy

$$\begin{aligned} E(t) &= \mathscr{A}(t, u) + \varepsilon \mathscr{F}(t, u) \\ &= \mathscr{E}(t) + (1+\varepsilon)\, \mathscr{F}(t, u) \\ &\leqslant (1+\varepsilon)\, \mathscr{F}(u) - \int_0^t \left[\mathscr{D}(\eta, u_\eta) + \mathscr{A}_t(\eta, u) - \mathscr{F}_t(\eta, u) \right] d\eta + \mathscr{E}(0) \equiv L(t), \end{aligned}$$

where ε is a positive constant which will be fixed later and where the third line is a consequence of the energy conservation law (2.5). Then by (1e)

$$\max\{ \mathscr{A}(t, u), \varepsilon \mathscr{F}(t, u) \} \leqslant L(t). \tag{4.3}$$

We show that the modified functional $L(t)$ has an "exponential" bound.

Indeed,

$$L'(t) = -\mathscr{D}(t, u_t) + \mathscr{A}_t(t, u) - \mathscr{F}_t(t, u) + (1+\varepsilon)\, \frac{d}{dt}\, \mathscr{F}(u)$$

and

$$\frac{d}{dt}\mathcal{F}(t, u(t)) = \langle F(t, u(t)), u_t(t)\rangle + \mathcal{F}_t(t, u(t)).$$

Now choose $\varepsilon \leqslant 1$ so small that $(1+\varepsilon)d_1 < 1$. The structure conditions (2e), (3e) yield

$$\begin{aligned}
L'(t) &= (1+\varepsilon)\langle F(t, u), u_t\rangle - \mathcal{D}(t, u_t) + \mathcal{A}_t(t, u) + \varepsilon\mathcal{F}_t(t, u)\\
&\leqslant [(1+\varepsilon)d_1 - 1]\,\mathcal{D}(t, u_t) + (1+\varepsilon)\,d_2\mathcal{F}(t, u)\\
&\qquad + (1+\varepsilon)d_3 + \mathcal{A}_t^+(t, u) + \mathcal{F}_t^+(t, u)\\
&\leqslant (1+\varepsilon)\,d_2\mathcal{F}(t, u) + (1+\varepsilon)d_3 + d_4(t)[\mathcal{A}(t, u) + \mathcal{F}(t, u)]\\
&\leqslant \left(\frac{(1+\varepsilon)}{\varepsilon}\,d_2 + \frac{1}{\varepsilon}\,d_4)\right) L(t) + (1+\varepsilon)d_3 \equiv \delta_2(t)\,L(t) + \delta_3(t)
\end{aligned}$$

by (4.3).

By the Gronwall lemma, we obtain

$$L(t) \leqslant L(0)\,\exp\left(\int_0^t \delta_2(s)\,ds\right) + \int_0^t \delta_3(s)\,\exp\left(\int_s^t \delta_2(\eta)\,d\eta\right) ds.$$

Using this estimate we see that $\mathcal{A}(t, u) + \mathcal{F}(t, u) \leqslant L(t)/\varepsilon$ and hence $\mathcal{A}(t, u) + \mathcal{F}(t, u)$ cannot become unbounded in finite time. The hypothesis H(i) asserts that local solutions exist for every $u_0 \in Z$. Then H(ii), together with this last estimate, asserts that such solutions can be continued to all of J and, consequently, every local solution can be extended to a global solution.

5. EXAMPLES

Let Ω be a bounded open set in R^n, $n \geqslant 1$, and $u: \Omega \times J \to R$. Consider the equation

$$|u_t|^{m-2}\,u_t - a\nabla\cdot(|\nabla u|^{q-2}\,\nabla u) = |u|^{p-2}\,u, \tag{5.1}$$

where a is a positive constant, subject to the homogeneous Dirichlet boundary conditions

$$u(x, t) = 0 \qquad \text{on} \quad \partial\Omega \times J \tag{5.2}$$

and the initial condition

$$u(x, 0) = u_0(x) \qquad \text{for} \quad x \in \Omega. \tag{5.3}$$

Then with $F(u) = |u|^{p-2} u$ and $A(u) = -a\nabla \cdot (|\nabla u|^{q-2} \nabla u)$, we have

$$\mathscr{F}(u) = \frac{1}{p} \int_\Omega |u|^p \, dx, \tag{5.4}$$

$$\mathscr{A}(u) = \frac{a}{q} \int_\Omega |\nabla u|^q \, dx \tag{5.5}$$

and $Q(t, y) = |y|^{m-2} y$. It is natural, moreover, to make the identification

$$\mathscr{D}(t, y) = \langle Q(y), y \rangle = \int_\Omega |y|^m \, dx \tag{5.6}$$

in the conservation law (2.5). We take $Y = \mathscr{L}^m(\Omega)$, $X = \mathscr{L}^p(\Omega)$, and $D = \mathscr{W}_0^{1,q}(\Omega)$.

First, suppose that

$$1 < m < p.$$

One checks that $X \subset Y$ and that conditions (a), (b) at the end of Section 2 hold with $\gamma = \delta = 1$. Consequently $\mathscr{D}$ satisfies (1d)–(3d). Moreover, hypothesis (1s) holds, while (2s) is also met if

$$1 < q < p.$$

Of course, (3s) is empty in the present autonomous case. Under these circumstances, there are clearly initial values $u_0 \in X \cap D \equiv Z$ for which $\mathscr{E}(0) < 0$, and consequently for which the solution of the initial–boundary value problem (5.1)–(5.3), even if it exists locally, cannot be global.

Now consider the hypotheses of Theorem 3. Suppose that the exponents m, p, q satisfy

$$1 < p \leqslant m, \qquad q > 1.$$

It follows that

$$\int_\Omega |u|^{p-2} uv \, dx \leqslant \left(\int_\Omega |v|^m \, dx \right)^{1/m} \left(\int_\Omega |u|^{(p-1)p'} \, dx \right)^{1/p'} \left(\int_\Omega 1^s \, dx \right)^{1/s}$$
$$\leqslant \frac{1}{m} \int_\Omega |v|^m \, dx + \frac{1}{p'} \int_\Omega |u|^p \, dx + \frac{1}{s} |\Omega|$$

by Young's inequality, where

$$\frac{1}{s}=\frac{1}{p}-\frac{1}{m}\geqslant 0.$$

Rewriting this using Eqs. (5.4), (5.5), and (5.6) yields

$$\langle F(u), v\rangle \leqslant \frac{1}{m}\,\mathscr{D}(t, v)+(p-1)\,\mathscr{F}(u)+\frac{m-p}{mp}\,|\Omega|.$$

This is precisely (4.1) with $d_1=1/m$, $d_2=p-1$ and $d_3=[(m-p)/mp\,|\Omega|$. Clearly (1e)–(3e) are satisfied. Now assume also that the hypothesis H(ii) holds for solutions of Eq. (5.1). (Whether this assumption indeed is valid is an open question. Moreover, as indicated in Remark 1, the satisfaction of this hypothesis in case $m=q=2$ requires at least that p be subcritical!). Solutions of Eq. (5.1) which exist locally in time must, in fact be continuable for all time and grow at most exponentially in time in the modified energy norm.

Finally, we consider an example that illustrates the comments in Remark 4. Consider the equation

$$|u|^{\kappa}\,|u_t|^{m-2}\,u_t - a\nabla\cdot(|\nabla u|^{q-2}\,\nabla u)=|u|^{p-2}\,u, \tag{5.7}$$

where a is a positive constant, subject to the homogeneous Dirichlet boundary conditions

$$u(x, t)=0 \qquad \text{on} \quad \partial\Omega\times J \tag{5.8}$$

and the initial condition

$$u(x, 0)=u_0(x) \qquad \text{for} \quad x\in\Omega, \tag{5.9}$$

where we now assume that $p>m$. Setting $Q(t, u, y)=|u|^{\kappa}\,|y|^{m-2}\,y$, we need to verify (3.14). Taking (with t suppressed)

$$\mathscr{D}(u, y)=\langle Q(t, u, y), y\rangle=\int_\Omega |u|^{\kappa}\,|y|^m\,dx, \tag{5.10}$$

an application of Hölder's inequality convinces us that (3.14) holds provided that $0\leqslant\kappa<p-m$. A related problem arises, however, in the definition of a solution. In particular, the requirement that $\mathscr{D}(u, u_t)$ be in $\mathscr{L}^1(J)$ does *not* force $u_t\in\mathscr{L}^m(\Omega)$, so that one can hardly suppose that $u\in AC(J\to\mathscr{L}^m(\Omega))$.

Rather it is appropriate to require instead that

$$|u|^{\kappa/m}\,u_t\in AC(J\to\mathscr{L}^m(\Omega)),$$

so that in particular

$$|u|^{\kappa/m}\, u_t \in \mathscr{L}^m(\Omega), \quad \text{a.a. } t,$$

and, in turn,

$$|u|^{\kappa}\, |u_t|^m \in \mathscr{L}^1(\Omega) \quad \text{a.a. } t.$$

This is exactly the condition that $\mathscr{D}(u, u_t)$ must be well defined (measurable) on J, in order for (2.5) to be meaningful.

The perturbed porous medium equation $w_t = \Delta(|w|^{l-1}\, w) + |w|^{s-1}\, w$, with $s > 1$ and $l > 0$, falls into the class of equations given by (5.7) if we set $u = |w|^{l-1} w$. Then $p = s/l + 1$, $\kappa = (1-l)/l$, and $q = m = 2$. In order to apply the result of the preceding paragraph, we must require $0 < (1-l)/l = \kappa \leqslant p - m = s/l - 1$ or $l < 1$ and $s > 1$. Thus, in the fast diffusion case, no local solution with negative initial energy (1.6) and with a time derivative $(u^{\kappa/2+1})_t \in \mathscr{L}^2(\Omega)$ can be global.

Although the results here do not give the global nonexistence result for the full range $1 < l < \infty$, with $s > \max\{l, 2\}$, they are sometimes applicable to equations which, at the outset, do not appear to be in the form dictated by (5.7). Also, just as in [7], it is possible to replace the powers $|w|^{l-1}\, w$ and $|w|^{s-1}\, w$ by functions $\beta(w)$ and $f(w)$, respectively, to obtain a similar result for more general nonlinearities. For example, the condition that replaces the pointwise version of $(2s)'$ is (with $q = 2$)

$$c_1 \int_0^{\beta(w)} f(s)\, d\beta(s) \leqslant c_2\, |\beta(w)|^p \leqslant \beta(w)\, f(w) - 2 \int_0^{\beta(w)} f(s)\, d\beta(s),$$

where we have assumed that β is monotone increasing.

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REFERENCES

1. J. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford (2)* **28** (1977), 473–486.
2. M. Fila, Boundedness of global solutions of nonlinear diffusion equations, *J. Differential Equations* **98** (1992), 226–240.
3. A. Friedman and B. McLeod, Blow-up of positive solutions of semilinear heat equations, *Indiana Univ. Math. J.* **34** (1985), 425–447.

4. V. A. Galaktionov, Boundary value problem for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta+1}$, *Differentsial'nye Uravneniya* **17**, No. 5 (1981), 836–842. [In Russian; English translation, *Differential Equations* **17** (1981), 551–555]
5. V. Georgiev and G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differential Equations* **109** (1994), 295–308.
6. S. Kaplan, On the growth of solutions of quasilinear parabolic equations, *Comm. Pure Appl. Math.* **16** (1957), 305–330.
7. T. Imai and M. Mochizuki, On blow-up of solutions for quasilinear degenerate parabolic equations, *Publ. Res. Inst. Math. Sci.* **27** (1991), 695–709.
8. A. A. Lacey and D. Tzanetis, Complete blow-up for a semilinear diffusion equation with a sufficiently large initial condition, *IMA J. Appl. Math.* **41** (1988), 207–215.
9. H. A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = [Au + \mathscr{F}(u)]$, *Arch. Rational Mech. Anal.* **51** (1973), 371–386.
10. H. A. Levine, The role of critical exponents in blow-up theorems, *SIAM Rev.* **32** (1990), 262–288.
11. H. A. Levine, Nonexistence of global solutions of nonlinear wave equations, *in* "Proceedings of N.S.F. Regional Conference on Improperly Posed Problems in Partial Differential Equations" (A. Carasso and E. P. Stone, Eds.), Pitman Research Notes in Mathematics, Vol. 1, pp. 95–104, Longman Sci. Math., Harlow, 1975.
12. H. A. Levine, P. Pucci, and J. Serrin, Some remarks on global nonexistence for nonautonomous abstract evolution equations, *Contemp. Math.* **208** (1997), 253–263.
13. H. A. Levine, and J. Serrin, Global nonexistence theorems for quasilinear evolution equations with dissipation, *Arch. Rational Mech. Anal.* **137** (1997), 341–361.
14. M. Nakao, Asymptotic stability for some nonlinear evolution equation of second order with unbounded dissipative terms, *J. Differential Equations* **30** (1978), 54–63.
15. P. Pucci and J. Serrin, Asymptotic stability for non-autonomous wave systems, *Comm. Pure Appl. Math.* **49** (1996), 177–216.
16. A. Samarskii, V. Galaktionov, S. Kurdyumov, and A. Mikhailov, "Blow-up in quasilinear parabolic equations," De Gruyter, Berlin, 1995.

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